JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 32, No. 4, July–August 2009

Static Formations Using Momentum Exchange Between Satellites

Steven G. Tragesser*
University of Colorado, Colorado Springs, Colorado 80933

DOI: 10.2514/1.40939

Feasible operations of satellite formations are dependent upon an efficient way to affect the relative motion. One possible means of formation stationkeeping is for satellites to exchange momentum in the form of radiation or mass. This approach could potentially provide a mutually repulsive force between satellites without the expenditure of propellant. This paper characterizes equilibria positions in the rotating orbital frame for an arbitrary number of satellites subjected to equal and opposite repulsive force between specifiable satellite pairs. Equilibrium configurations are found that allow arbitrary dispersion in the plane perpendicular to nadir. Stability analysis of these equilibria show stable out-of-plane motion and unstable in-plane motion. A strategy is developed to stabilize the system and to reconfigure the formation geometry. Finally, a simulation including Earth oblateness demonstrates the dynamic feasibility of this method of orbit control.

Nomenclature

d = nominal distance between satellites
K_D = derivative gain for out-of-plane damping
K_p = proportional gain for in-plane control
m_i = mass of ith satellite
n = mean motion of the reference orbit
$\bar{R}_i$ = position of the ith satellite in an Earth-centered inertial frame
$\bar{r}_i$ = position of the ith satellite relative to a circular reference orbit
$\bar{r}_{ij}$ = position of the ith satellite relative to the jth satellite ($\bar{r}_{ij} = \bar{r}_i - \bar{r}_j$)
$\bar{u}_{ij}$ = repulsive force between ith and jth satellite ($= -\bar{u}_{ji}$)
u_{ix} = x component of control acceleration on the ith satellite
u_{iy} = y component of control acceleration on the ith satellite
x_i = radial coordinate for ith satellite
x_i^* = equilibrium (nominal) value of radial coordinate for ith satellite
y_i = along-track coordinate for ith satellite
y_i^* = equilibrium (nominal) value of along-track coordinate for ith satellite
z_i = out-of-plane coordinate for ith satellite
z_i^* = equilibrium (nominal) value of out-of-plane coordinate for ith satellite
α_{ij} = magnitude of the repulsive force between the ith and jth satellite
Δz_i = nominal separation distance between the ith and $(i+1)$th satellites ($\Delta z_i = z_i - z_{i+1}$)
δx_i = radial displacement from equilibrium for ith satellite
δy_i = along-track displacement from equilibrium for ith satellite
δz_i = out-of-plane displacement from equilibrium for ith satellite

Presented as Paper 7509 at the AIAA Astrodynamics Specialist Conference, Honolulu, HI, 17–21 August 2008; received 11 September 2008; revision received 7 January 2009; accepted for publication 1 April 2009.

*Assistant Professor, Mechanical and Aerospace Engineering, 1420 Austin Bluffs Parkway; stragesser@eas.uccs.edu. Senior Member AIAA.

I. Introduction

SATELLITE formations show such promise for enhanced performance of missions in space that researchers are arduously researching relative control using conventional thrusting [1], tethers [2–4], Lorentz forces [5], coulomb forces from charged satellites [6–8], and the linear impulse from lasers [9–11]. Theoretically, all of these stationkeeping approaches except for thrusting can provide some formation control without expenditure of fuel. Instead, replenishable electrical energy is used, allowing for very long mission durations.

Bae has investigated the use of the momentum from lasers for relative control of satellites [9,10]. In his papers, he proposes sending a laser beam back and forth between two satellites using mirrors. The momentum transfer from the photon stream creates a repulsive force. The addition of an attractive force using tethers creates a nearly rigid link between satellites, allowing for a great variety of geometric possibilities for the formation. The repulsive force of Bae's concept could also potentially be attained by a stream of mass traveling between the satellites [11], although this concept has not been explored in depth in the literature. Here, equal and opposite mass streams impart an equal and opposite linear impulse. Equal flow rates between the satellites would maintain equal mass balance throughout the system. Both of these means of achieving a repulsive force have one satellite creating an impulse of momentum which is then captured by the other satellite and redirected toward the first. This method of generating a repulsive force for orbit control will be referred to as "momentum exchange."

Momentum exchange through either lasers or mass streams has many technical difficulties, including the accurate pointing and dispersion of the momentum stream. For mass exchange, there are the additional complications of satellite contamination and the evaporation/sublimation of the stream. The exact mechanism of momentum exchange is not considered in this paper, but the orbital equations below are relevant for either radiation or mass exchange provided mass flow 1) is small compared to the mass of the spacecraft and 2) transits the distance between the satellites on a timescale much smaller than the orbit period. Mass flow through a flexible, enclosed conduit is yet another possible exchange mechanism, but the dynamics of this system is beyond the scope of this paper.

Tethered, coulomb, and momentum exchange systems are interesting complements to one another for formation flying applications. Tethers provide strictly attractive forces, whereas momentum exchange provides a strictly repulsive force. Equilibrium configurations for tethered objects have been studied for two-, three- [2], and four-body [3] systems, but in all cases the static configuration is confined to the orbital plane. (An apparent exception to this limitation occurs with tetrahedral tethered formations in [4], but the authors model the tethers as rigid rods capable of compressive forces.) Conversely, this paper shows that momentum exchange is capable of achieving arbitrary dispersions perpendicular to the orbital plane. Momentum exchange, therefore, is an attractive design possibility for Earth-oriented remote sensing, particularly when a dispersion of satellites perpendicular to the observation line of sight is desired.

Coulomb forces can be either attractive or repulsive, so they are capable of reproducing the dynamic behavior of either tethers or momentum exchange for any two-satellite formation. Indeed, two-satellite formations are possible along any of the Hill frame axes, and this design has been developed extensively by Berryman and Schaub [6] and Natarajan and Schaub [7]. However, with coulomb forces, all satellites must interact, whereas tethers and momentum exchange allow interactions to be controlled individually. Berryman and Schaub [6] analyzed linear and triangular three-craft formations and developed equations that determine nominal charges for an arbitrary number of craft, but formation design and control becomes very complex for a large number of spacecraft. Conversely, this paper shows that complex geometries are available via momentum exchange with each satellite interacting with at most two other satellites, greatly simplifying the design and control of highly populated formations.

This paper provides the first dynamic characterization of an arbitrary number of satellites subjected to equal and opposite repulsive forces between specifiable satellites. Equilibrium configurations are determined for a static formation; that is, all spacecraft maintain a fixed relative location with respect to a reference frame that rotates with the mean motion of the system center of mass. The stability and reconfiguration of these formations are considered.

II. Static Equilibrium for Mutually Repulsive Forces

A. Governing Equations of Motion

The governing equations for the motion relative to a circular reference orbit of the ith satellite of a formation subject to a mutually repulsive force with $N-1$ other formation members are [12]

$$m_i(\ddot{x}_i - 2n\dot{y}_i - 3n^2x_i) = \sum_{j=1}^{N} \bar{u}_{ij}\cdot\hat{x}$$
$$m_i(\ddot{y}_i + 2n\dot{x}_i) = \sum_{j=1}^{N} \bar{u}_{ij}\cdot\hat{y}$$
$$m_i(\ddot{z}_i + n^2z_i) = \sum_{j=1}^{N} \bar{u}_{ij}\cdot\hat{z} \tag{1}$$

where m_i is the mass of the ith satellite and x_i, y_i, and z_i are the coordinates along the rectilinear radial, tangential, and normal directions, respectively, of the circular reference orbit, shown in Fig. 1, with mean motion n. The right-hand side of these relative motion equations represents the force due to momentum exchange along each of the coordinate axes. This force is along the line of action between two satellites and is given by

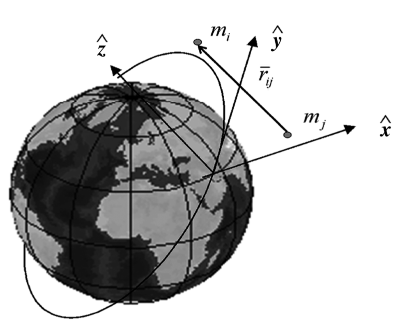


Fig. 1 Orbital reference frame and two-craft *z*-axis formation.

$$\bar{u}_{ij} = \alpha_{ij}\frac{\bar{r}_{ij}}{|\bar{r}_{ij}|}, \qquad \alpha_{ij} > 0 \tag{2}$$

where α_{ij} is the magnitude of the force between the ith and jth satellites. This parameter must be positive for a repulsive force between bodies. The satellite masses are assumed constant, which would be valid after the formation is established (assuming minimal losses) or if the momentum exchange streams have negligible mass. The orbital mechanics of the momentum streams themselves are neglected; that is, the streams are assumed to move along the line of action between the satellites. This assumption is valid in the case of photon streams or mass streams that transit the intersatellite distance in much less than an orbital period.

Equilibrium solutions (with respect to the orbiting reference frame) can be obtained by setting all time derivatives equal to zero:

$$3m_in^2x_i^* + \sum_{j=1}^{N}\alpha_{ij}\frac{(x_i^*-x_j^*)}{|\bar{r}_{ij}^*|} = 0$$
$$\sum_{j=1}^{N}\alpha_{ij}\frac{(y_i^*-y_j^*)}{|\bar{r}_{ij}^*|} = 0$$
$$m_in^2z_i^* - \sum_{j=1}^{N}\alpha_{ij}\frac{(z_i^*-z_j^*)}{|\bar{r}_{ij}^*|} = 0 \tag{3}$$

where the superscript asterisk denotes nominal equilibrium states.

B. Two-Craft Formations

We consider solutions to Eqs. (3) for various configurations of formations composed of two satellites. Configurations are considered for satellite placements along each of the three reference frame axes of Fig. 1 and along an arbitrary direction.

1. *Displacement Along the X Axis*

Assuming displacements between two bodies along the x axis, reduces Eqs. (3) to

$$3m_1n^2x_1^* + \alpha_{12}\frac{(x_1^*-x_2^*)}{|x_1^*-x_2^*|} = 0 \quad 3m_2n^2x_2^* + \alpha_{12}\frac{(x_2^*-x_1^*)}{|x_2^*-x_1^*|} = 0 \tag{4}$$

Solving these equations restricts the system center of mass to the origin of the reference orbit:

$$x_1^* = -\frac{m_2}{m_1}x_2^* \tag{5}$$

The value of either x_1^* or x_2^* can be chosen arbitrarily, with the other specified by the preceding equation. Without loss of generality, assume $x_1^* > 0$. Solving either of Eqs. (4) for the magnitude of the applied force yields

$$\alpha_{12} = -3m_1n^2x_1^* \tag{6}$$

This result, however, contradicts the stipulation in Eq. (2) that the momentum exchange force must always have a positive coefficient (i.e., be repulsive). Therefore, this case is not possible for a momentum exchange system. A tensile force, such as that of a space tether or a coulomb formation, would be capable of this equilibrium condition. The preceding applied force is the well-known gravity-gradient tension found in a tether [13] and is also directly comparable to the charge determined in [6] for this type of configuration.

2. *Displacement Along the Y Axis*

Assuming displacements only along the y axis for two bodies reduces Eqs. (3) to

$$\alpha_{12}\frac{(y_1^*-y_2^*)}{|y_1^*-y_2^*|} = 0 \qquad \alpha_{12}\frac{(y_2^*-y_1^*)}{|y_1^*-y_2^*|} = 0 \tag{7}$$

The solution

$$\alpha_{12} = 0 \tag{8}$$

shows that an equilibrium exists only in the absence of any momentum exchange. This result corresponds to the well-known leader–follower formation.

3. Displacement Along the Z Axis

As with the x axis, the equilibrium conditions specify that the center of mass be collocated with the origin of the reference orbit:

$$z_1^* = -\frac{m_2}{m_1} z_2^* \tag{9}$$

With one of the satellite locations chosen arbitrarily (e.g., to specify mission requirements), this equation specifies the location of the second satellite. Alternatively, both locations can be specified, with Eq. (9) determining the satellite mass ratio. Without loss of generality, choose $z_1^* > 0$. Solving the equilibrium conditions for the magnitude of the applied force yields

$$\alpha_{12} = m_1 n^2 z_1^* = -m_2 n^2 z_2^* \tag{10}$$

For this case, α_{12} takes on a positive value, demonstrating the feasibility of this configuration for a mutually repulsive force. For a low-Earth-orbit formation (altitudes between 160 and 2000 km), this results in a momentum exchange rate of 1 μN per kilogram of satellite mass per meter of separation. For geostationary orbit (GEO) this drops to 0.005 μN per kilogram per meter. Thrust levels of 35 μN have been demonstrated with a laser [14], indicating potential feasibility of this concept for several hundred kilogram GEO satellites with a separation distance of several meters.

4. Displacement Along an Arbitrary Direction

For a displacement between two satellites that is not along the coordinate axes, Eqs. (3) become

$$\begin{aligned}
&3m_1 n^2 x_1^* + \alpha_{12}\frac{(x_1^* - x_2^*)}{|\bar{r}_{12}|} = 0\\
&3m_2 n^2 x_2^* + \alpha_{12}\frac{(x_2^* - x_1^*)}{|\bar{r}_{12}|} = 0\\
&\alpha_{12}\frac{(y_1^* - y_2^*)}{|\bar{r}_{12}|} = 0\\
&\alpha_{12}\frac{(y_2^* - y_1^*)}{|\bar{r}_{12}|} = 0\\
&m_1 n^2 z_1^* - \alpha_{12}\frac{(z_1^* - z_2^*)}{|\bar{r}_{12}|} = 0\\
&m_2 n^2 z_2^* - \alpha_{12}\frac{(z_2^* - z_1^*)}{|\bar{r}_{12}|} = 0
\end{aligned} \tag{11}$$

The following conditions are necessary for $\alpha_{12} > 0$:

$$x_1^* = x_2^* = 0 \qquad y_1^* = y_2^* \tag{12}$$

Therefore, the equilibrium solution can have no displacement along the x axis. Furthermore, there can be no difference in the displacement along the y axis, although the exact location along the y axis is arbitrary. Therefore, the only equilibrium solution for two satellites with a nonzero mutually repulsive force is a displacement strictly along the z axis, with the center of mass located at an arbitrary y-axis location. This formation is depicted in Fig. 1.

5. Multiple Pairs of Satellites with Displacement Along the Z Axis at Arbitrary Y-Axis Locations

Members of a propellant exchange formation do not have to interact (whereas, in a coulomb formation, all members have a charge that affects all other members). As a result, multiple independent pairs of two-craft formations can make up a much larger formation. As shown in the previous section, each pair of satellites can be located at an arbitrary along-track location. Therefore, multiple pairs of satellites dispersed in the along-track direction can form various geometries in the y-z plane. This geometry can be depicted as a discrete representation of two functions with the y-axis (along-track) location as the abscissa and the z-axis (cross-track) location as the ordinate. Both of these functions can be chosen arbitrarily with the mass ratio determined by Eq. (9). Figure 2 shows three formations composed of multiple satellite pairs. The formation members are indicated by black dots. The radius of these dots is proportional to the mass of each of the formation members. In Fig. 2a, a circular function is used to determine the locations of the upper formation elements. A function symmetric about the y axis is used to specify the locations of the lower members. By Eq. (9), upper and lower masses must be equal. The force required by momentum exchange is proportional to the length of the arrows, where the linear dependence on the z-axis displacement is evident. In Fig. 2b, the z-axis location of the upper and lower members are chosen randomly. The masses of the upper members are all equal, whereas the masses of the lower members are determined by Eq. (9). Figure 2c shows a function that is generated using circular functions of varying radii to generate a more complete aperture.

Although the functions used to define the nominal locations of the satellites do not have restrictions on smoothness, they are functions in the strict mathematical sense. That is, only one satellite and its pair can be located at any particular along-track location to keep the momentum exchange streams from colliding with other satellites. The following sections investigate the possibility of satellites using multiple momentum exchange streams to build formations that do not rely strictly on satellite pairs.

C. Collinear Three-Craft Formations Along the Z Axis

The conclusion of Sec. II.B.4 is easily extended to three (or more) spacecraft. That is, no x-axis displacement or difference in y-axis displacement is physically feasible for an equilibrium solution. Therefore, only a collinear formation along the z axis (at an arbitrary y-axis location) is possible for three (or more) satellites. We will further assume that momentum exchange only occurs between neighboring satellites along the z axis. The z-axis displacements and the magnitude of the repulsive force are governed by the equilibrium equations:

$$\begin{aligned}
&m_1 n^2 z_1^* - \alpha_{12}\frac{(z_1^* - z_2^*)}{|z_1^* - z_2^*|} = 0\\
&m_2 n^2 z_2^* - \alpha_{12}\frac{(z_2^* - z_1^*)}{|z_2^* - z_3^*|} - \alpha_{23}\frac{(z_2^* - z_3^*)}{|z_2^* - z_3^*|} = 0\\
&m_3 n^2 z_3^* - \alpha_{23}\frac{(z_3^* - z_2^*)}{|z_3^* - z_2^*|} = 0
\end{aligned} \tag{13}$$

Without loss of generality assume $z_1^* > z_2^* > z_3^*$, yielding

$$\begin{aligned}
&m_1 n^2 z_1^* - \alpha_{12} = 0\\
&m_2 n^2 z_2^* + \alpha_{12} - \alpha_{23} = 0\\
&m_3 n^2 z_3^* + \alpha_{23} = 0
\end{aligned} \tag{14}$$

Suppose we specify the outermost location for the satellites z_1^* and z_3^*. Solving Eqs. (14) then gives the familiar center of mass condition (specifying the location of the middle satellite), along with the required momentum exchange forces to equilibrate the system:

$$\begin{aligned}
&z_2^* = \frac{1}{m_2}(-m_1 z_1^* - m_3 z_3^*)\\
&\alpha_{12} = m_1 n^2 z_1^*\\
&\alpha_{23} = -m_3 n^2 z_3^*
\end{aligned} \tag{15}$$

Thus, the collinear formation along the z axis is feasible as long as the highest mass position z_1^* is positive and the lowest mass position z_3^* is negative to satisfy the center of mass condition and result in a positive coefficient for the momentum exchange.

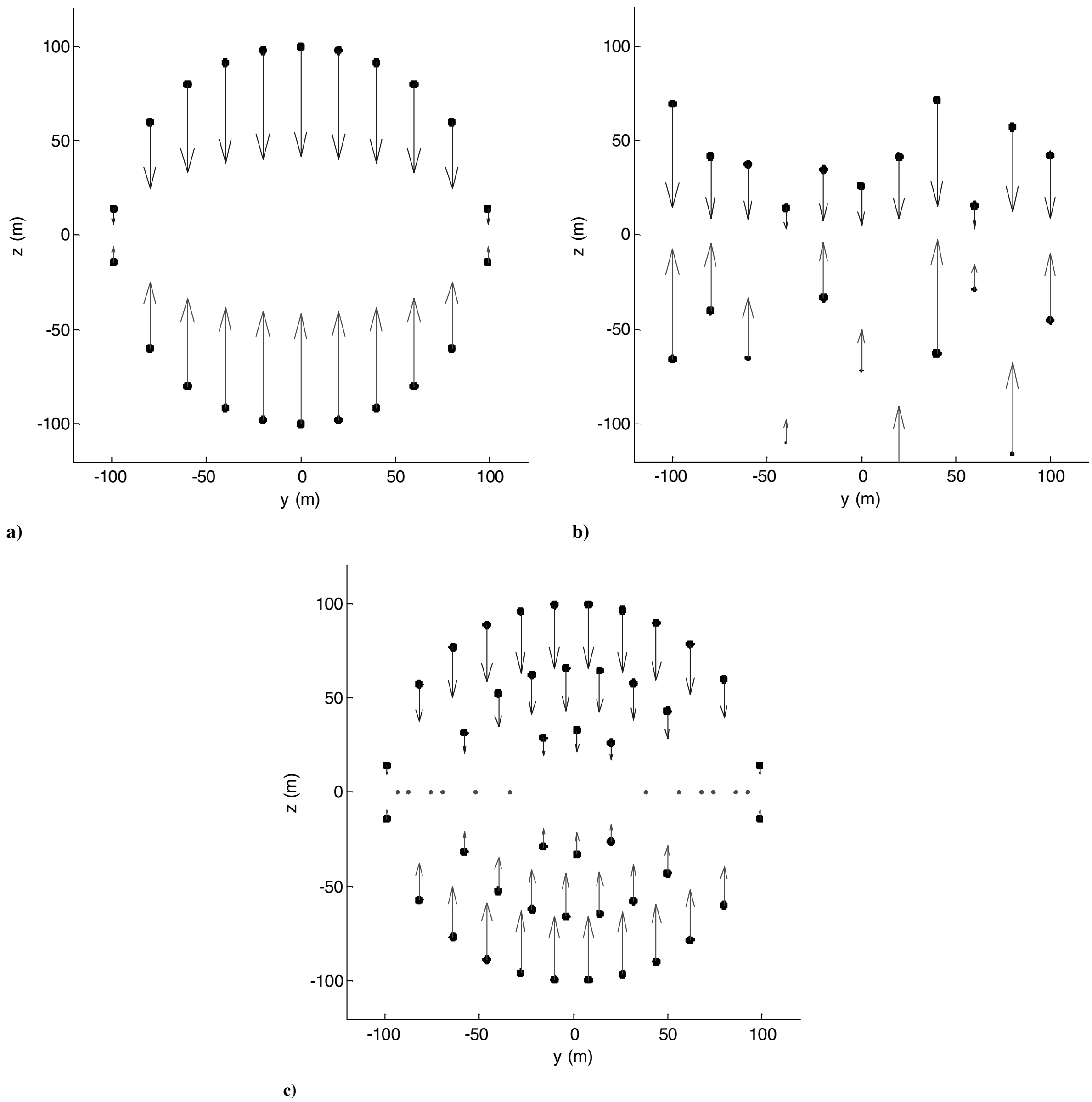


Fig. 2 Formations of multiple two-craft pairs in the plane perpendicular to nadir: a) circular, b) function of random numbers, and c) function generated from multiple circular functions. Arrow length is proportional to momentum exchange rate and circle size is proportional to mass.

D. Collinear *N*-Craft Formations Along the *Z* Axis

The collinear formation of three satellites can be extended to an arbitrary number of satellites with each satellite exchanging momentum with the satellites on either side. Other solutions, where momentum exchange is shared between satellites that are not neighboring, are mathematically possible but will not be considered here due to the practical concerns of beam interference and dispersion. Without loss of generality, assume $z_1^* > z_2^* > \cdots > z_{N-1}^* > z_N^*$ to yield

$$
\begin{aligned}
&m_1 n^2 z_1^* - \alpha_{12} = 0\\
&m_2 n^2 z_2^* + \alpha_{12} - \alpha_{23} = 0\\
&m_3 n^2 z_3^* + \alpha_{23} - \alpha_{34} = 0\\
&\vdots\\
&m_{N-1} n^2 z_{N-1}^* + \alpha_{N-2,N-1} - \alpha_{N-1,N} = 0\\
&m_N n^2 z_N^* + \alpha_{N-1,N} = 0
\end{aligned}
\tag{16}
$$

Various approaches could be employed to solve these N equations. Here, we assume that the satellite masses are specified. The locations of $N-1$ satellites can then be specified, with the location of the Nth satellite and the $N-1$ momentum exchange magnitudes determined by Eqs. (16). However, mission objectives are most likely dependent upon relative separations between satellites, rather than their absolute position in space. Therefore, we specify the geometry in terms of the $N-1$ separation distances:

$$\Delta z_i \equiv z_i - z_{i+1}, \qquad i = 1, \ldots, N-1 \tag{17}$$

Then, solving Eqs. (16) determines the z coordinate for each satellite and the magnitude of the momentum exchange for each pair of satellites:

$$
\begin{aligned}
&z_1^* = \frac{1}{m_{\text{total}}}\left(\sum_{i=2}^{n} m_i \sum_{j=1}^{i-1} \Delta z_j\right)\\
&z_i = z_{i-1} + \Delta z_{i-1}, \qquad i = 2, \ldots, n\\
&\alpha_{12} = m_1 n^2 z_1^*\\
&\alpha_{23} = m_2 n^2 z_2^* + \alpha_{12}\\
&\vdots\\
&\alpha_{N-1,N} = m_{N-1} n^2 z_{N-1}^* + \alpha_{N-2,N-1}
\end{aligned}
\tag{18}
$$

Each satellite is a member of two satellite pairs (i.e., is exchanging momentum with two other satellites) except for the first and Nth satellites. Figure 3 shows a formation composed of five subformations of five equal mass satellites, each with a separation distance of 50 m. Satellites closer to the y axis must provide a greater amount of momentum exchange because they are ultimately "supporting" all of the satellites above and below.

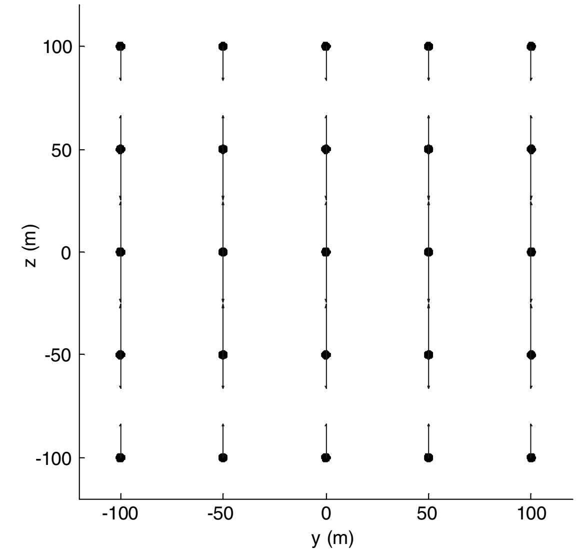


Fig. 3 Formation with five collinear subformations each composed of five equal mass satellites shown as black dots. Formation plane is perpendicular to nadir. Arrow length is proportional to momentum exchange rate.

III. Stability of Collinear Z-Axis Formations

A. Two-Craft Formations

We determine if the two-craft formation along the z axis is Lyapunov stable. The momentum exchange forcing function in the equations of motion in Eq. (1) is nonlinear. We linearize this forcing function about the equilibrium condition for the two satellites:

$$\begin{aligned} z_1 &= z_1^*, \qquad y_1 = y_1^*, \qquad x_1 = \dot{x}_1 = \dot{y}_1 = \dot{z}_1 = 0 \\ z_2 &= z_2^*, \qquad y_2 = y_2^*, \qquad x_2 = \dot{x}_2 = \dot{y}_2 = \dot{z}_2 = 0 \end{aligned} \tag{19}$$

where $y_1^* = y_2^*$. Using a binomial series expansion and neglecting higher-order terms, the denominator of the right-hand side becomes

$$\begin{aligned} \frac{1}{|\bar{r}_1 - \bar{r}_2|} &= [(\delta x_1 - \delta x_2)^2 + (\delta y_1 - \delta y_2)^2 \\ &\quad + (z_1^* + \delta z_1 - z_2^* - \delta z_2)^2]^{-1/2} \\ &= \frac{1}{d}\left[1 + \frac{2}{d}(\delta z_1 - \delta z_2) + \cdots\right]^{-1/2} \\ &\approx \frac{1}{d}\left[1 - \frac{1}{d}(\delta z_1 - \delta z_2)\right] \end{aligned} \tag{20}$$

where $d = z_1^* - z_2^*$ is the separation distance between satellites and $z_1^* > z_2^*$ is assumed. Substituting Eq. (20) into Eqs. (1) and eliminating higher-order terms yields the governing linear equations

$$\begin{aligned} \delta\ddot{x}_1 - 2n\delta\dot{y}_1 - 3n^2\delta x_1 &= \frac{\alpha_{12}}{m_1 d}(\delta x_1 - \delta x_2) \\ \delta\ddot{y}_1 + 2n\delta\dot{x}_1 &= \frac{\alpha_{12}}{m_1 d}(\delta y_1 - \delta y_2) \\ \delta\ddot{z}_1 + n^2(z_1^* + \delta z_1) &= \frac{\alpha_{12}}{m_1} \\ \delta\ddot{x}_2 - 2n\delta\dot{y}_2 - 3n^2\delta x_2 &= \frac{\alpha_{12}}{m_2 d}(\delta x_2 - \delta x_1) \\ \delta\ddot{y}_2 + 2n\delta\dot{x}_2 &= \frac{\alpha_{12}}{m_2 d}(\delta y_2 - \delta y_1) \\ \delta\ddot{z}_2 + n^2(z_2^* + \delta z_2) &= -\frac{\alpha_{12}}{m_2} \end{aligned} \tag{21}$$

The z equations have decoupled from the in-plane coordinates. These equations are simple harmonic oscillators with constant forcing functions. The general solution is given by

$$\delta z_i = A\cos nt + B\sin nt - (-1)^i\frac{\alpha_{12}}{m_i n^2} - z_i^* \tag{22}$$

Substituting the nominal momentum exchange from Eq. (10) and applying initial conditions yields

$$\delta z_i = \delta z_i(0)\cos nt + \frac{\delta\dot{z}_i(0)}{n}\sin nt \tag{23}$$

Thus, the out-of-plane coordinate oscillates with mean zero and is indeed Lyapunov stable. The z motion could be made asymptotically stable by adding a rate-feedback term to the momentum exchange. We will develop this concept further in a later section on reconfiguration maneuvers.

To analyze the stability of the in-plane coordinates, we write the equations of motion in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \delta\ddot{x}_1 \\ \delta\ddot{y}_1 \\ \delta\ddot{x}_2 \\ \delta\ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 0 & -2n & 0 & 0 \\ 2n & 0 & 0 & 0 \\ 0 & 0 & 0 & -2n \\ 0 & 0 & 2n & 0 \end{bmatrix}\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{y}_1 \\ \delta\dot{x}_2 \\ \delta\dot{y}_2 \end{bmatrix} + \begin{bmatrix} -3n^2 - \psi_1 & 0 & \psi_1 & 0 \\ 0 & -\psi_1 & 0 & \psi_1 \\ \psi_2 & 0 & -3n^2 - \psi_2 & 0 \\ 0 & \psi_2 & 0 & -\psi_2 \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta y_1 \\ \delta x_2 \\ \delta y_2 \end{bmatrix} = 0 \tag{24}$$

where

$$\psi_1 \equiv \frac{\alpha_{12}}{m_1 d} = \frac{z_1 n^2}{d}, \qquad \psi_2 \equiv \frac{\alpha_{12}}{m_2 d} = \frac{z_2 n^2}{d} \tag{25}$$

Using the identity

$$\psi_1 + \psi_2 \equiv \frac{n^2}{d}(z_1 + z_2) = n^2 \tag{26}$$

the characteristic polynomial for this system can be simplified to

$$s^8 + 3n^4 s^4 + 4n^6 s^2 = 0 \tag{27}$$

This equation is independent of the mass ratio of the two bodies (i.e., not a function of ψ_1 or ψ_2). After eliminating the zero roots, this equation is cubic in s^2 and can be solved analytically. The roots for this system are 0, 0, $\pm ni$, $(-1.12 \pm 0.87i)n$, $(1.12 \pm 0.87i)n$. The positive real part in the roots indicates unstable behavior.

Natarajan and Schaub determine that charge modulation can be used to stabilize a radially aligned coulomb formation by exploiting gravity-gradient torques [7]. Unfortunately, stabilization in this manner is not possible for the out-of-plane (z-axis aligned) formation considered in this paper. Therefore, we resort to thruster control. Assuming a control acceleration proportional to the displacement from the nominal position and equal gains for both x- and y-motion yields

$$u_{ix} = K_p\delta x_i, \qquad u_{iy} = K_p\delta y_i \tag{28}$$

As an alternative to controlling the absolute positions, the y component could control the relative displacement between the two satellites in the along-track direction. This approach may reduce the required control effort, but will allow the entire formation to drift. The control in Eq. (28) yields the following linear equations of motion:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\delta\ddot{x}_1\\\delta\ddot{y}_1\\\delta\ddot{x}_2\\\delta\ddot{y}_2\end{bmatrix}+\begin{bmatrix}0&-2n&0&0\\2n&0&0&0\\0&0&0&-2n\\0&0&2n&0\end{bmatrix}\begin{bmatrix}\delta\dot{x}_1\\\delta\dot{y}_1\\\delta\dot{x}_2\\\delta\dot{y}_2\end{bmatrix}$$
$$+\begin{bmatrix}-3n^2-\psi_1+K_p&0&\psi_1&0\\0&-\psi_1+K_p&0&\psi_1\\\psi_2&0&-3n^2-\psi_2+K_p&0\\0&\psi_2&0&-\psi_2+K_p\end{bmatrix}\times\begin{bmatrix}\delta x_1\\\delta y_1\\\delta x_2\\\delta y_2\end{bmatrix}=0 \tag{29}$$

This system has an eighth-order characteristic polynomial with only even terms:

$$s^8+a_6s^6+a_4s^4+a_2s^2+a_0=0 \tag{30}$$

where the coefficients are

$$a_6=4K_p \qquad a_4=6K_p^2-8n^2K_p+3n^4$$
$$a_2=4K_p^3-16n^2K_p^2+6n^4K_p+4n^6$$
$$a_0=K_p^4-8n^2K_p^3+19n^4K_p^2-12n^6K_p \tag{31}$$

Including a derivative feedback term yields all nonzero coefficients, permitting asymptotic stability. The application of Routh's necessary conditions for stability to these nonlinear coefficients in K_p results in only one region for which the system is stable:

$$K_p>4n^2 \tag{32}$$

This equation specifies the smallest proportional gain for stabilization, which will in turn require smaller control effort. Performing a similar analysis for a free-flying satellite without momentum exchange yields a minimum gain of $3n^2$. Therefore, momentum exchange does incur a small additional cost to stabilize the in-plane motion, but this is small compared to the propellant saved for the z displacement.

B. *N*-Craft Formations

As with the two-craft formation, linearizing the equations of motion for an arbitrary number of satellites decouples the in-plane and out-of-plane motion. For the out-of-plane motion,

$$\delta\ddot{z}_1+n^2\delta z_1=\frac{\alpha_{12}}{m_1}-n^2z_1^*$$
$$\delta\ddot{z}_2+n^2\delta z_2=\frac{\alpha_{23}}{m_2}-\frac{\alpha_{12}}{m_2}-n^2z_2^*$$
$$\vdots$$
$$\delta\ddot{z}_{N-1}+n^2\delta z_{N-1}=\frac{\alpha_{N-1,N}}{m_{N-1}}-\frac{\alpha_{N-2,N-1}}{m_{N-1}}-n^2z_{N-1}^*$$
$$\delta\ddot{z}_N+n^2\delta z_N=-\frac{\alpha_{N-1,N}}{m_N}-n^2z_N^* \tag{33}$$

Using the equilibrium conditions for the magnitude of the momentum exchange from Eq. (18), the right-hand side of these equations vanishes, resulting in a decoupled system of harmonic oscillators. Therefore, the Lyapunov-stable z motion of the two-craft formation is preserved for an arbitrary number of satellites.

As in the two-craft case, the x-y motion of N satellites displaced along the z axis is unstable and must be stabilized by feedback control. The gain required for stability increases as the number of satellites increases. For a three-craft formation, the linearized in-plane equations of motion become

$$\delta\ddot{x}_1-2n\delta\dot{y}_1-3n^2\delta x_1=\frac{\alpha_{12}}{m_1d_1}(\delta x_1-\delta x_2)-K_p\delta x_1$$
$$\delta\ddot{y}_1+2n\delta\dot{x}_1=\frac{\alpha_{12}}{m_1d_1}(\delta y_1-\delta y_2)-K_p\delta y_1$$
$$\delta\ddot{x}_2-2n\delta\dot{y}_2-3n^2\delta x_2=\frac{\alpha_{12}}{m_2d_1}(\delta x_2-\delta x_1)+\frac{\alpha_{23}}{m_2d_2}(\delta x_2-\delta x_3)-K_p\delta x_2$$
$$\delta\ddot{y}_2+2n\delta\dot{x}_2=\frac{\alpha_{12}}{m_2d_1}(\delta y_2-\delta y_1)+\frac{\alpha_{23}}{m_2d_2}(\delta y_2-\delta y_3)-K_p\delta y_2$$
$$\delta\ddot{x}_3-2n\delta\dot{y}_3-3n^2\delta x_3=\frac{\alpha_{23}}{m_3d_2}(\delta x_3-\delta x_2)-K_p\delta x_3$$
$$\delta\ddot{y}_3+2n\delta\dot{x}_3=\frac{\alpha_{23}}{m_3d_2}(\delta y_3-\delta y_2)-K_p\delta y_3 \tag{34}$$

where $d_1=z_1^*-z_2^*$ and $d_2=z_2^*-z_3^*$ are the nominal distances between satellites. Unlike the two-craft case, the characteristic polynomial for this system is dependent on the mass ratios between the satellites. The resulting equation is very long and difficult to use for any analytic determination of the gain for stability. If the masses of the three satellites are assumed to be equal, then the equation reduces significantly to

$$s^{12}+a_{10}s^{10}+a_8s^8+a_6s^6+a_4s^4+a_2s^2+a_0=0 \tag{35}$$

where the coefficients are

$$a_{10}=6K_p-5n^2$$
$$a_8=15K_p^2-37n^2K_p+21n^4$$
$$a_6=20K_p^3-98n^2K_p^2+124n^4K_p-11n^6$$
$$a_4=15K_p^4-122n^2K_p^3+294n^4K_p^2-205n^6K_p+34n^8$$
$$a_2=6K_p^5-73n^2K_p^4+300n^4K_p^3-457n^6K_p^2+132n^8K_p+72n^{10}$$
$$a_0=K_p^6-17n^2K_p^5+109n^4K_p^4-327n^6K_p^3+450n^8K_p^2-216n^{10}K_p \tag{36}$$

Applying Routh's criteria provides the following sufficient condition for stability for the three-craft formation:

$$K_p>6n^2 \tag{37}$$

Applying this analysis to larger numbers of satellites (assuming equal masses) reveals a general closed-form expression that specifies a sufficient condition for stability for an arbitrary number of satellites:

$$K_p>\left(6-N+\sum_{i=3}^{N}i\right)n^2 \tag{38}$$

IV. Reconfiguration

The nominal z displacement of the satellites can be reconfigured by modulating the amplitude of the momentum exchange. Suppose we want to move satellite 1 from an initial nominal location of z_{1o}^* to a final nominal value of z_{1f}^*. A control law of the form

$$\alpha_{12}(t)=-2m_1nK_D\dot{z}_1+m_1n^2z_{1f}^* \tag{39}$$

yields a second-order response. To illustrate, substitute Eq. (39) into the equation of motion for the z axis [the third equation in Eq. (1)] and assume small displacements in x_i and y_i:

$$\ddot{z}_1 + 2nK_D\dot{z}_1 + n^2 z_1 = n^2 z_{1f}^* \tag{40}$$

This is a second-order oscillator with damping K_D and particular solution z_{1f}^*:

$$z_1(t) = e^{-K_D nt}\left(-\Delta z_1^* \cos\omega_d t - \frac{\Delta z_1^* K_D n}{\omega_d}\sin\omega_d t\right) + z_{1f}^* \tag{41}$$

where $\Delta z_1^* = z_{1f}^* - z_{1o}^*$, $\omega_d = n\sqrt{1-K_D^2}$, and the satellite is assumed to start at rest. Design of the control in Eq. (39) is achieved by choosing the damping coefficient to affect a desired response.

Even though the position of the second satellite is not included as feedback into the momentum exchange term, this satellite also displays a second-order response with damping K_D. Taking the derivative of Eq. (41) and substituting it into Eq. (1) yields the equation of motion for the second satellite:

$$\ddot{z}_2 + n^2 z_2 = -2\frac{m_1}{m_2}\frac{\Delta z_1^* n^2}{\omega_d} nK_D e^{-K_D nt}\sin\omega_d t - \frac{m_1}{m_2}n^2 z_{1f}^* \tag{42}$$

Although there is no damping term on the left-hand side of this equation, the coefficients for the homogenous part of the solution are zero after initial conditions are applied, resulting in a damped second-order response due to the forcing function. The solution to this ordinary differential equation is

$$z_2(t) = \frac{m_1}{m_2}nK_D e^{-K_D nt}\left(\frac{\Delta z_1^*}{K_D n}\cos\omega_d t + \frac{\Delta z_1^*}{\omega_d}\sin\omega_d t\right) - \frac{m_1}{m_2}z_{1f}^* \tag{43}$$

with the same damping as the first satellite and a particular solution of z_{2f}^*.

V. Results

To validate these results, several simulations are performed for the momentum exchange concept. To derive the relative motion in Eq. (1), a linear approximation was employed. The simulation does not make such an approximation; instead, the nonlinear equations of motion are used:

$$\ddot{\bar{R}}_i = -\frac{\mu}{R_i^3}\bar{R}_i + \bar{J}_2 + \bar{u}_i \tag{44}$$

where $\bar{J}_2$ is the perturbing acceleration due to Earth oblateness and $\bar{u}_i$ indicates the perturbing acceleration due to both momentum exchange and thruster control as described earlier. The masses for the two satellites are 1000 and 100 kg for m_1 and m_2, respectively. The orbit radius of the reference is 6800 km. Making use of Eq. (9), the nominal positions with respect to this reference are

$$y_1^* = y_2^* = 10 \text{ m}, \qquad z_1^* = 10 \text{ m}, \qquad z_2^* = -100 \text{ m} \tag{45}$$

From Eq. (10), the momentum exchange rate for these equilibrium positions is 0.013 N. The satellites are initially perturbed from this nominal by a small amount:

$$\delta x_1(0) = 0, \qquad \delta y_1(0) = 0.01 \text{ m}, \qquad \delta z_1(0) = 0.01 \text{ m}$$
$$\delta x_2(0) = 0.01 \text{ m}, \qquad \delta y_2(0) = 0, \qquad \delta z_2(0) = 0 \tag{46}$$

The minimum gain for a stable system for this example is $K_p > 4n^2 = 5.1 \times 10^{-6}$ as described in Eq. (32). A simulation is performed for a feedback gain of $K_p = 4.01n^2$. The resulting motion is shown in Fig. 4 for 30 orbits without oblateness in the simulation. Figure 4a shows the trajectory of both masses projected onto the orbital plane, and Fig. 4b depicts the associated control effort. This case has periodic behavior indicative of a neutrally stable system. The motion and control effort for satellite 1 have an amplitude that is about an

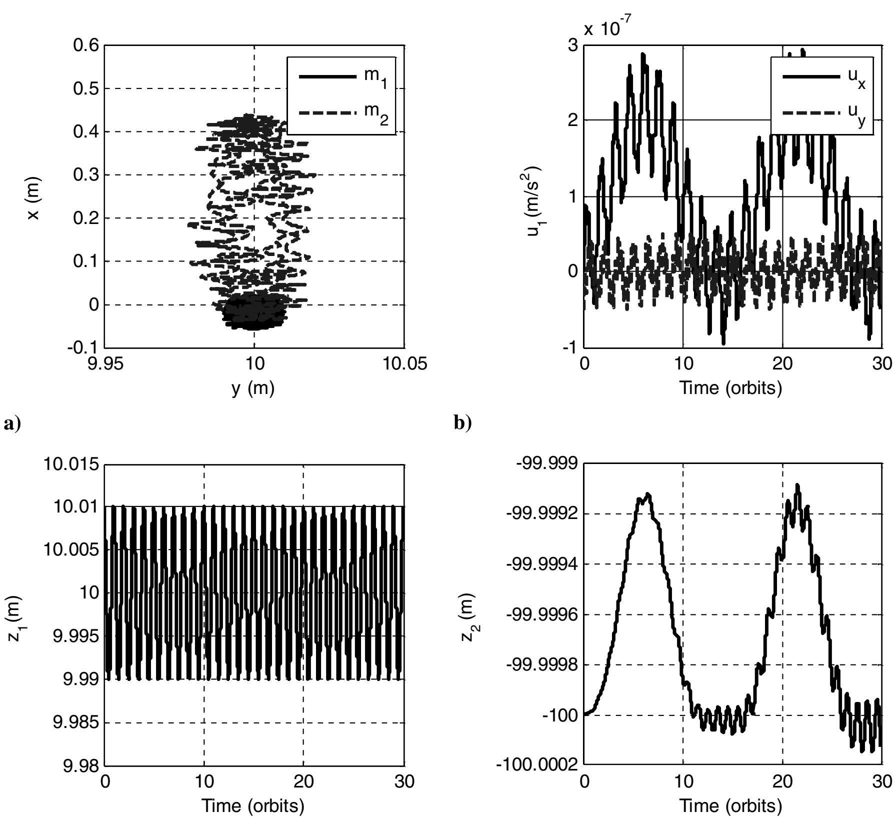


Fig. 4 Simulation results for stable case with $K_p = 4.01n^2$: a) in-plane motion, b) control effort on m_1, c) out-of-plane motion of m_1, and d) out-of-plane motion of m_2.

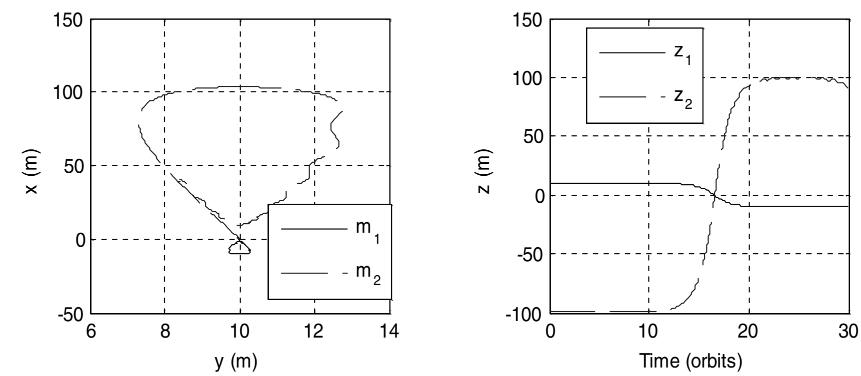
order of magnitude less than satellite 2, corresponding to the difference in the masses (and hence the difference in the nominal z displacement). Figures 4c and 4d show the stable z-axis motion. The displacement for satellite 1 is an oscillation with magnitude equal to the initial perturbation of 0.01 m. The coupled dynamics induce a small oscillation in satellite 2, even though there was no initial disturbance along the z axis.

Figure 5 shows the simulation results for the unstable feedback gain of $3.99n^2$. This small change in the gain has a dramatic effect on trajectory. Unstable behavior is clear in both the in-plane and out-of-plane trajectories. The z-axis motion switches between the original equilibrium of $z_1^* = 10$ m, $z_2^* = -100$ m and the only other equilibrium solution (for these two masses and momentum exchange rate) of $z_1^* = -10$ m, $z_2^* = 100$ m.

For satellites with similar aerodynamic profiles, oblateness will probably be the main perturbative effect on the relative dynamics. The result of adding the J_2 term into the equations of motion is shown in Fig. 6. The amplitudes of both the in-plane and out-of-plane excursions from the nominal have significantly increased (as compared with Fig. 4), but the system still exhibits Lyapunov-stable behavior. The out-of-plane motion of the upper satellite (Fig. 6b) reaches a maximum deviation of 0.07 m away from the nominal value of 10 m. The z motion, however, is uncontrolled. Damping could easily be introduced into the momentum exchange term as described for the reconfiguration maneuver.

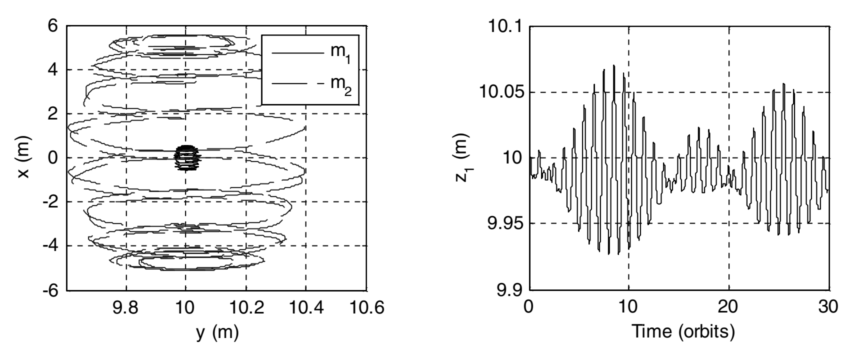
The magnitude of the oscillation in the motion can be reduced by increasing the gain. The gain was increased from the barely stable value of $4.01n^2$ to $8n^2$. The results are shown in Fig. 7. In-plane deviations from the nominal equilibrium are reduced by an order of magnitude; maximum out-of-plane deviations are reduced by a fifth.

To determine if the analytic conclusions for larger formations are valid for a higher-fidelity model, the motion of five satellites was



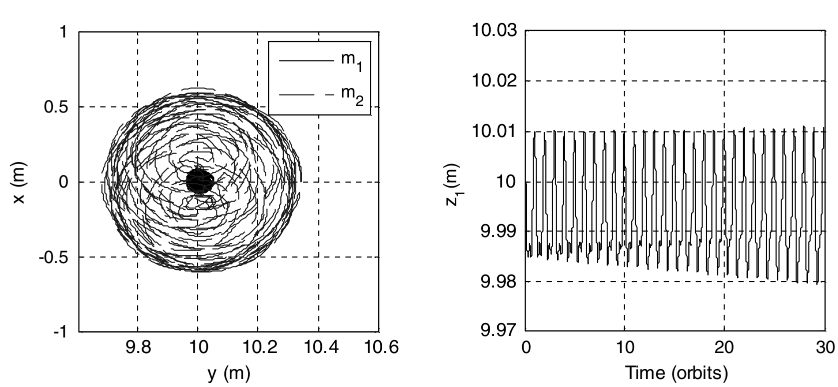
a) b)

Fig. 5 Simulation results for unstable case with $K_p = 3.99n^2$: a) in-plane motion, and b) out-of-plane motion.



a) b)

Fig. 6 Simulation results for stable case with oblateness: a) in-plane motion, and b) out-of-plane motion.



a) b)

Fig. 7 Simulation results with oblateness and increased gain showing reduced deviation: a) in-plane motion, and b) out-of-plane motion.

simulated. The nominal values for the five-craft formation were chosen as

$$y_1^* = y_2^* = y_3^* = y_4^* = y_5^* = 10 \text{ m} \qquad z_1^* = 20 \text{ m}, \quad z_2^* = 10 \text{ m}$$
$$z_3^* = 0 \text{ m}, \quad z_4^* = -10 \text{ m}, \quad z_5^* = -20 \text{ m} \tag{47}$$

The initial perturbations for satellites 1 and 2 are the same as for the previous case, and the other three satellites are initially unperturbed. All masses are assumed equal to 1000 kg. From Eq. (38), the condition for in-plane stability is $K_p > 13n^2$. Figures 8 and 9 show the in-plane and out-of-plane projections of the trajectory for gains that bracket this condition. Figure 8 depicts the stable behavior of the system for $K_p = 13.01n^2$. The z motion is shown as a bold dash-dot line to distinguish it from the plot grid lines. The small amplitude of the in-plane motion indicates that the collinear formation stays within 0.3 deg of the nominal orientation parallel to the z axis. Figure 9, on the other hand, shows unstable behavior associated with the gain $K_p = 12.99n^2$.

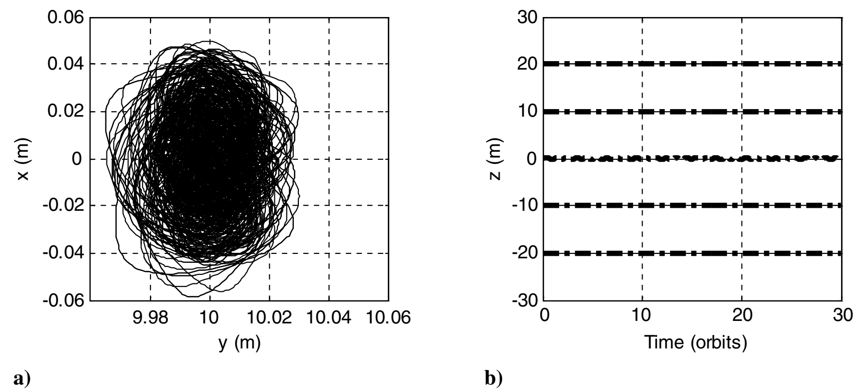


Fig. 8 Stable trajectory with oblateness for five-craft formation ($K_p = 13.01n^2$): a) in-plane motion, and b) out-of-plane motion.

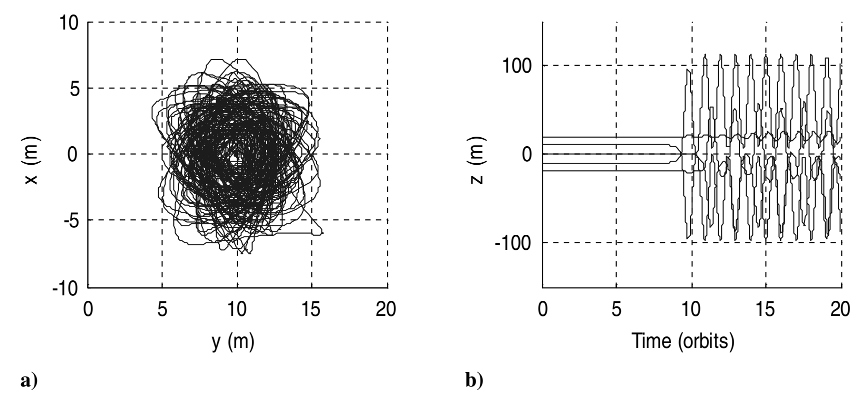


Fig. 9 Unstable trajectory with oblateness for five-craft formation ($K_p = 12.99n^2$): a) in-plane motion, and b) out-of-plane motion.

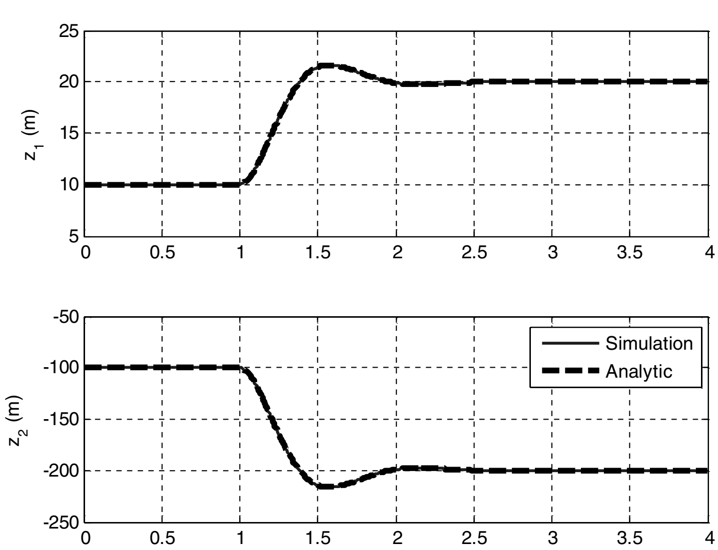


Fig. 10 Reconfiguration maneuver showing second-order response.

We now consider a reconfiguration maneuver. The satellites are initialized in the first two-craft example, but, after one orbit, the satellites are maneuvered to new nominal z displacements of $z_1^* = 20$ m and $z_2^* = -200$ m. The damping coefficient K_D is chosen to be 0.5 to give moderate overshoot with good response time. The resulting motion along the z axis is plotted in Fig. 10 using the nonlinear simulation with oblateness. The analytic solution to the motion is included on the plots and is not discernibly different from the simulation. The maximum difference between the two plots is 0.06 m for satellite 1 and 0.15 m for satellite 2. These numbers will vary with different initial perturbations, because these are not incorporated into the analytic solution.

VI. Conclusions

The mutually repulsive force available through momentum exchange permits equilibria of an arbitrary number of bodies along the axis perpendicular to the orbital plane. These collinear arrays can also be dispersed in the along-track direction to design formations of satellites with arbitrary dispersion in the plane perpendicular to nadir. The motion of these satellites perpendicular to the orbit plane is stable, but the in-plane motion is unstable and requires some alternate method of stabilization. Momentum exchange could be combined with tethers to provide a link with nearly rigid characteristics. This concept may permit practical design and operation of more complicated, three-dimensional formations.

Acknowledgments

This work was sponsored by the U.S. Air Force Research Laboratory, Propulsion Directorate, Edwards Air Force Base, California. The author also wishes to thank Andrew Ketsdever for general discussions on the momentum exchange concept.

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